MATH 254 – PRACTICE FINAL

(1) Determine whether the following statements are true or false. Circle T for true, F for false. You DO NOT need to justify your answers.

- (a) T F If A and B are both orthogonal, then so is AB.
- (b) T F For any matrix A, AA^T is always symmetric.
- (c) T F Let A be $n \times n$ matrix. A is invertible if and only if $dim(row(A)) = n$.
- (d) T F If a square matrix A has 0 as an eigenvalue, then A is singular.
- (e) T F Every orthonormal set of n vectors in $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$.
- (f) T F A and A^T have the same eigenvalues and the same eigenvectors.
- (g) T F Let $T : \mathbb{R}^n \to \mathbb{R}^n$ and $rank(T) = n - 1$. Then T is one-to-one.
- (h) T F If A is $n \times n$ matrix, and λ is an eigenvalue of A, then the system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has only trivial solution.
- (i) T F If A is diagonalizable, then A is invertible.
- (j) T F If $T : V \to W$ is a linear transformation, then $nullity(T) \leq dim(V)$.

(2) Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

(a) Find all eigenvalues of A.
(b) Find a basis for eigenspaces of A.
(c) Clearly state why A is diagonalizable. Find a matrix P that diagonalizes A.
(d) Find eigenvalues and bases for eigenspace of A^8.

(3) Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $L(x, y, z) = (x + z, -x + y + z)$.

(a) Show that L is a linear transformation.
(b) Find the standard matrix A of L.
(c) Find bases for $range(L)$ and $ker(L)$.
(d) Is L one-to-one? Is L onto? Explain?

(4) Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $L(x, y, z) = (x + z, -x + y + z)$, with $S = \{(1, 1, 1),\ (0, -1, 0),\ (0, 0, -1)\}$ and $T = \{(1, 1),\ (-2, 0)\}$ as bases for $\mathbb{R}^3$ and $\mathbb{R}^2$; respectively.

(a) Find the matrix A of L with respect to S and T.
(b) Let $\mathbf{w} = (1, 2, 3)$. Compute $[\mathbf{w}]_S$.
(c) Compute $[L(\mathbf{w})]_T$.

(5) Show that characteristic equation of a 2×2 matrix A can be expressed as

$$\lambda^2 - tr(A)\lambda + det(A) = 0.$$

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(6) Suppose that A is diagonalizable. Prove that $A + kI_n$ is diagonalizable for any scalar k.

(7) Is the set of all orthogonal matrices a subspace of M_{mn}? If yes, prove it. If no, provide example to show why it is not a subspace.

(8) **Please go over homework assignments #12 to #13**.

(9) Go over test 1, test 2, and test 3.

(10) Go over practice 1, practice 2, and practice 3.

Check answers

(1) (a) T, (b) T, (c) T, (d) T, (e) T, (f) F, (g) F, (h) F, (i) F, (j) T.

(2) (a) $det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & -1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 1) = 0 \implies \lambda = 1, -1.$

(b) $\underline{\lambda = 1}$: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \implies \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \begin{matrix} x_1 = s \\ x_2 = -t \\ x_3 = t \end{matrix}$

Thus basis for eigenvectors of $(\lambda = 1)$ is $\{(1, 0, 0), (0, -1, 1)\}$
Similarly, we find basis for eigenvector of $\lambda = -1$ is $\{(0, 1, 1)\}$.

(c) Since the 3×3 matrix A has three linearly independent eigenvectors, A is diagonalizable.
$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(d) Eigenvalues of A^8 is λ^8 with the same corresponding eigenvectors. Thus, eigenvalue of A^8 is $(\pm 1)^8 = 1$ and the basis for eigenvectors is $\{(-1, 1, 0),\ (0, 0, 1),\ (1, 1, 0)\}$

(3) (a) Let $\mathbf{x}_1 = (x_1, y_1, z_1)$ and $\mathbf{x}_2 = (x_2, y_2, z_2)$ be vectors in $\mathbb{R}^3$. Then

$$\begin{aligned} L(\mathbf{x}_1 + \mathbf{x}_2) &= ((x_1 + x_2) + (z_1 + z_2),\ -(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2)) \\ &= (x_1 + z_1, -x_1 + y_1 + z_1) + (x_2 + z_2, -x_2 + y_2 + z_2) = L(\mathbf{x}_1) + L(\mathbf{x}_2), \\ L(c\mathbf{x}_1) &= (cx_1 + cz_1,\ -cx_1 + cy_1 + cz_1) = cL(\mathbf{x}_1). \end{aligned}$$

Thus, L is a linear transformation.

(b) Standard matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

(c) $rref(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Thus, $range(L) = col(A) = \{(1, -1), (0, 1)\}$ and $ker(L) = nullspace(A)$.

$$A\mathbf{x} = 0 \implies \begin{matrix} x_1 = -x_3 \\ x_2 = -2x_3 \end{matrix} \implies \begin{matrix} x_1 = -t \\ x_2 = -2t \\ x_3 = t \end{matrix} \implies null(A) = ker(L) = \{(-1, -2, 1)\}$$

(d) L is not one-to-one since $ker(L) \neq 0$. However, L is onto since dim(range(L))=dim($\mathbb{R}^2$)=2.

(4) (a) First we map the basis S to $\mathbb{R}^2$:

$$L(1,1,1) = (2,1), \quad L(0,-1,0) = (0,-1), \quad L(0,0,-1) = (-1,-1).$$

Then we need to write these images in term of basis T:

$$\begin{bmatrix} 1 & -2 & | & 2 & 0 & -1 \\ 1 & 0 & | & 1 & -1 & -1 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & | & 1 & -1 & -1 \\ 0 & 1 & | & -1/2 & -1/2 & 0 \end{bmatrix}$$

Thus, the matrix A of L with respect to S and T is $\begin{bmatrix} 1 & -1 & -1 \\ -1/2 & -1/2 & 0 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 1 & -1 & 0 & | & 2 \\ 1 & 0 & -1 & | & 3 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & -2 \end{bmatrix}$. Thus, $[\mathbf{w}]_S = [1\ -1\ -2]^T$.

(c) $[L(\mathbf{w})]_T = A[\mathbf{w}]_S = [4\ 0]^T$.

(5) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Characteristic equation for A is

$$|A - \lambda I| = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad - bc) = \lambda^2 - Tr(A)\lambda + det(A) = 0.$$

(6) Since A is diagonalizable, there exists an invertible matrix P such that $P^{-1}AP = D$ for some diagonal matrix D. Note that

$$P^{-1}(A + kI_n)P = P^{-1}AP + P^{-1}(kI)P = D + kP^{-1}IP = D + kI.$$

$D+kI$ is diagonal matrix since it is a sum of two diagonal matrices. Hence, $A+kI$ is diagonalizable and it is also diagonalized by P.

(7) Recall: A is an orthogonal matrix if $AA^T = I_n$. Let A and B be orthgonal matrices, then

$$\begin{aligned} (A+B)(A+B)^T &= (A+B)(A^T + B^T) = AA^T + AB^T + BA^T + BB^T \\ &= 2I_n + AB^T + BA^T \neq I_n. \end{aligned}$$

Thus, the set of orthogonal matrices is not closed under addition, which implies it is not a subspace of M_{mn}.