Def.: Solution of a system of linear equations is a set of numbers which when substituted into the set of variables, satisfies every equations in the system. The set of all solutions is called solution set of the system.

Ex.1: $x_1 + 6x_2 - 2x_3 = 4 \implies x_1 = 4 - 6x_2 + 2x_3$ has solution $x_1 = 4 - 6s + 2t,\ x_2 = s,\ x_3 = t$. Solution set is $\{(r,s,t)\big| r = 4 - 6s + 2t\}$.

Ex.2: $\begin{cases} x_1 + x_2 &= 7 \\ x_1 - x_2 &= 7 \end{cases}$ has solution $S = \{(7,0)\}$.

Ex.3: $\begin{cases} x_1 + x_2 &= 7 \\ x_1 - x_2 &= 7 \\ 2x_2 - 1 &= 0 \end{cases}$ has no solution; i.e., $S = \emptyset$

Def.: A consistent system of linear equations has at least one solution. An inconsistent system has no solution.

Ex.1 and 2 are consistent. Ex 3 is inconsistent.

Theorem: Every linear system has no solution, exactly one solution, or infinitely many solutions.

Def.: Augmented matrix for a system of linear equation is a matrix of coefficients adjoining the constant on the right hand side.

Ex 4

$$\begin{array}{llll} x_1 & +6x_2 & -2x_3 & = 4 \\ x_1 & & +x_3 & = 7 \\ x_1 & -2x_2 & +x_3 & = -1 \end{array} \implies \begin{bmatrix} 1 & 6 & -2 & \vdots & 4 \\ 1 & 0 & 1 & \vdots & 7 \\ 1 & -2 & 1 & \vdots & -1 \end{bmatrix}$$

Solve the system to get:

$$\begin{array}{ll} x_1 & = -2 \\ x_2 & = 4 \\ x_3 & = 9 \end{array} \implies \begin{bmatrix} 1 & 0 & 0 & \vdots & -2 \\ 0 & 1 & 0 & \vdots & 4 \\ 0 & 0 & 1 & \vdots & 9 \end{bmatrix}$$

Elementary row operations:

(1) Operation # 1: Interchange 2 rows; denoted by $R_i \longleftrightarrow R_j$.
(2) Operation # 2: IMultiply a row by a non-zero constant; denoted by kR_i.
(3) Operation # 3: IMultiply a row by a non-zero constant and add to another row; denoted by $kR_i + R_j \longrightarrow R_j$.

Reduce Row-Echelon form:

- First entry of a non-zero row is a 1, called leading 1.
- Zero row, if there is any, is at the bottom.
- In any successive non-zero rows, leading 1 in the lower row occurs further to the right than the one in the higher row.
- Each column that contains a leading 1 has zero everywhere else.

If a matrix satisfies only the first three criteria, then it is in row-echelon form.

Ex 5

$$\begin{bmatrix} 0 & 1 & 2 & : & 1 \\ 0 & 0 & 1 & : & 1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \implies \begin{aligned} x_2 = 1 - 2x_3 = -1 \\ x_3 = 1 \\ x_1 = t \text{ (free variable)} \end{aligned}$$

Gaussian/Gauss-Jordan Elimination:

- make first entry in the first row becoming the leading 1 (using 1st and/or 3rd operation)
- Use 3rd operation to make all entries below the leading 1 becoming zero.
- Repeat the process for 2nd row and then 3rd row.
- We will get a matrix with 1's in diagonal and zeros below these 1's. Now start from the bottom row and work backward: use 3rd operation to make all entries above the 1's zeros.

Note: One can vary steps to avoid fractions.

Homogeneous System:: system of linear equation in which all constant terms are zero.

Ex 6

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

$$rref \begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \begin{aligned} x_1 &= -2t \\ x_2 &= t \\ x_3 &= t \end{aligned}$$

Properties of Homogeneous Systems:

- Homogeneous system always has trivial solution $\implies$ consistent.
- Homogeneous system can has one solution (trivial) or infinitely many solutions (trivial and non-trivial).
- A homogeneous system with more unknowns than equation will have infinitely many solutions.

More examples::

Ex.7 Solve the system of equations $\begin{cases} 2x + y = a \\ 3x + 6y = b \end{cases}$

Ex.8 Consider the system $\begin{cases} x - ay = 0 \\ bx + 2y = 0 \end{cases}$.

For what values of a and b does the system has one solution, no solution, or infinitely many solutions?

Ex.9 Indicate whether the statement is always true or sometimes false. Justify your answer with logical argument or a counterexample.

(1) If a matrix is reduced to r.r.e.f. by two different sequences of elementary row operations, the resulting matrices will be different.

(2) A linear system of 3 equations with 5 unknowns must be consistent.

(3) A linear system of 5 equations with 3 unknowns must be inconsistent.

(4) If the r.r.e.f. of the augmented matrix has a row of zero, then the system must has infinitely many solutions.