Def.: Matrix is an rectangular array of numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Entry of matrix: a_{ij}: i^{th} row, j^{th} column.
Matrix A has size $m \times n$; that is, A has m rows and n columns.

Special matrices:

- Square matrix: matrix with size $n \times n$.
- Zero matrix: matrix where all entries are zero.
- Identity matrix I_n: an $n \times n$ matrix with $a_{ij} = 0$ if $i \neq j$ and $a_{ij} = 1$ if $i = j$.
- Diagonal matrix: an $n \times n$ matrix that has $a_{ij} = 0$ if $i \neq j$.
- Upper/lower triangular matrix: square matrix with zero entries in lower/upper main diagonal.

Operations on matrices:

(1) Equality: $A = B$ if they have the same size and corresponding entries are equal.

(2) Add/sub.: add or subtract corresponding entries of *same size* matrices.
Properties:

$$A + B = B + A,\ A + (B + C) = (A + B) + C,\ A + \mathbf{0} = A,\ A + (-A) = \mathbf{0}.$$

(3) Multiply by scalar k: multiply each entry by k.
Properties:

$$r(sA) = (rs)A,\ (r + s)A = rA + sA,\ r(A + B) = rA + rB,\ A(rB) = r(AB)$$

(4) Transpose: $[a_{ij}]^T = [a_{ji}]$; i.e., interchanging row and column of A.
Properties:

$$(A^T)^T = A, \qquad (A \pm B)^T = A^T \pm B^T, \qquad (rA)^T = rA^T, \qquad (AB)^T = B^T A^T$$

Def. A is a *symmetric* matrix if $A^T = A$.
This means A has to be a square matrix with $a_{ij} = a_{ji}$.
Def. A is a *skew symmetric* matrix if $A^T = -A$.
This means A has to be a square matrix with $a_{ij} = -a_{ji}$.

(5) Trace of a square matrix A, denoted by $Tr(A)$, is the sum of main diagonal entries.

(6) Dot product: Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$.

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

Ex.: $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ -1 \\ 3 \end{bmatrix} = (1)(4) + (2)(0) + (3)(-1) + (4)(3) = 13.$

(7) Matrix multiplication: Let A be $m \times p$ and B be $p \times n$ matrices, then the product $AB = C$ is an $m \times n$ matrix where each entry

$$c_{ij} = row_i(A) \cdot col_j(B)$$

(8) matrix to a power: Let A be a square matrix and r, s are integers.

$$A^r = \underbrace{AA \cdots A}_{r}$$

$$A^r A^s = A^{r+s}$$

$$(A^r)^s = A^{rs}$$

Properties of matrix multiplications:

- Identity matrix acts as multiplicative identity: $I_n A = A I_n = A$
- Multiply by $\mathbf{0}$ matrix: $\mathbf{0}A = \mathbf{0}$ and $A\mathbf{0} = \mathbf{0}$
- Let A, B, and C be matrix with approriated size.

$$A(BC) = (AB)C, \quad A(B + C) = AB + AC, \quad (A + B)C = AC + BC$$

- In general, $AB \neq BA$
- $AC = BC$ does NOT imply $A = B$.
- $AB = 0$ does NOT imply $A = 0$ or $B = 0$.

Ex: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$ $D = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$
We can easily verify that

$$AB = AC \text{ but } B \neq C$$

$$AD = 0 \text{ but } A \neq 0, \ D \neq 0$$

Examples:

Ex.1: Show that for any square matrix M, $M + M^T$ is symmetric.

Ex.2: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 1 \\ -1 & 2 \\ 1 & 0 \\ 2 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 6 \\ -2 & 1 & 3 & 0 \end{bmatrix}$$

Compute (if possible) $2A + B^T, \ C^2, \ ED, \ DE, \ Tr(BA)$.

Ex.3: Prove: If AB and BA are both defined, then AB and BA are square matrices.

Ex.4: True or false: AA^T and $A^T A$ is always a square matrix.

Ex.5: True or false: $Tr(AA^T) = Tr(A^T A)$.