Def.: Let A be a square matrix. If there is a matrix B of same size such that $AB = BA = I$, then A is said to be invertible (nonsingular) and B is called the inverse of $\mathbf{A}$; denoted $B = A^{-1}$.

Properties: Suppose that A and B are $n \times n$ invertible matrices. Then

(1) Inverse is unique. (So we can denote inverse by A^{-1})
Proof: Suppose that C and D are both inverses of A, then $CA = AD = I$. Thus,

$$C = C(AD) = (CA)D = D$$

(2) A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.
Proof: $AA^{-1} = I$ and $A^{-1}A = I$ means A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(3) AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
Proof: $(AB)(AB)^{-1} = I \implies B(AB)^{-1} = A^{-1} \implies (AB)^{-1} = B^{-1}A^{-1}$

(4) A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T$.
Proof: $AA^{-1} = I \implies (AA^{-1})^T = I^T \implies (A^{-1})^T A^T = I \implies (A^T)^{-1} = (A^{-1})^T$.

(5) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, then $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
Proof: verify that $AA^{-1} = I$.

(6) For any nonzero scalar k, kA is invertible and $(kA)^{-1} = \dfrac{1}{k}A^{-1}$.
Proof: see page 45 in text.

(7) For any positive interger k, A^k is invertible and $(A^k)^{-1} = (A^{-1})^k$.

Examples:

Ex.1: Find the inverse of

$$A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Ex.2: Let $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$. Compute A^{-3} and $(A^T)^{-1}$.

Ex.3: Prove that if A is nonsingular and $AB = AC$, then $B = C$.

Ex.4: Assume that A and B and $n \times n$ matrices. State whether the following statement is true or false.

(a) $(AB)^2 = A^2B^2$ $\quad$ (b) $(AB^{-1})(BA^{-1}) = I_n$ $\quad$ (c) $(A + B)^2 = A^2 + 2AB + B^2$