(1) **Elementary matrix:** An $n \times n$ matrix is called elementary matrix if it can be obtained from the identity matrix $\mathbf{I}_n$ by performing a single elementary row operations.
Ex:

$$\mathbf{E}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ are elementary. } C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ is not elementary.}$$

(2) **Properties:**

(a) When a matrix $\mathbf{A}$ is multiplied on the *left* by an elementary matrix $\mathbf{E}$, the result is the same as performing the same row operation on $\mathbf{A}$.
Thus, for every matrix A, there exist a sequence of elementary matrices $E_1, E_2, \cdots, E_k$ such that

$$E_k \cdots E_2 E_1 A = rref(A).$$

Ex:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & -1 & 0 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \implies E_1 A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 0 & -3 & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \implies E_2 E_1 A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies E_3 E_2 E_1 A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & -1 \end{bmatrix}$$

We can continue the process until we get $rref(A)$.

(b) Elementary matrix is invertible and its inverse is also an elementary matrix.

(3) **Theorem:** Let A be $n \times n$ matrix. The following statements are equivalence:

(a) A is invertible.
(b) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
(c) $A\mathbf{x} = \mathbf{0}$ has only trivial solution.
(d) The $rref(A)$ is I_n
(e) A is expressible as a product of elementary matrices.

(4) **Row equivalence:** If a matrix B can be obtained from a matrix A by performing a finite sequence of row operations, then A and B are row equivalence.

(5) **Finding A^{-1}:** From part (d) in theorem above, A is invertible iff it is row equivalent to I_n. Therefore, if A is invertible, we have

$$E_k \cdots E_3 E_2 E_1 A = I_n \implies E_k \cdots E_3 E_2 E_1 = I_n A^{-1} \implies A^{-1} = E_k \cdots E_3 E_2 E_1 I_n.$$

This means we can perform a sequence of row operations on A to obtain I_n and do the same row of operation on I_n to get A^{-1}. Thus, to find A^{-1}, we do

$$\left[\mathbf{A} \;\vdots\; I_n\right] \cdots \text{ row operations } \cdots \left[I_n \;\vdots\; A^{-1}\right]$$

If the *left* side has a row of zeros; i.e., does not look like I_n, then A does not have inverse.

Ex.:

$$\begin{bmatrix} 1 & 2 & 0 & : & 1 & 0 & 0 \\ 0 & 3 & -1 & : & 0 & 1 & 0 \\ 1 & -1 & 0 & : & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & : & 1 & 0 & 0 \\ 0 & 3 & -1 & : & 0 & 1 & 0 \\ 0 & -3 & 0 & : & -1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & : & 1 & 0 & 0 \\ 0 & 0 & -1 & : & -1 & 1 & 1 \\ 0 & -3 & 0 & : & -1 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 0 & : & 1 & 0 & 0 \\ 0 & -3 & 0 & : & -1 & 0 & 1 \\ 0 & 0 & -1 & : & -1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & : & 1/3 & 0 & 2/3 \\ 0 & 1 & 0 & : & 1/3 & 0 & -1/3 \\ 0 & 0 & 1 & : & 1 & -1 & -1 \end{bmatrix}$$

(6) **Linear system and invertability:**

(a) Suppose that $\mathbf{A}$ is an invertible matrix. Then the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution; namely, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

Note: this only applies to systems that have number of equations equals to number of variables.

Ex:

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_2 - x_3 &= -3 \\ x_1 - x_2 &= 4 \end{aligned} \implies A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & -1 & 0 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

- Solve by Gauss-Jordan elimination:

$$\begin{bmatrix} 1 & 2 & 0 & : & 1 \\ 0 & 3 & -1 & : & -3 \\ 1 & -1 & 0 & : & 4 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & -1 \\ 0 & 0 & 1 & : & 0 \end{bmatrix} \implies x_1 = 3,\ x_2 = -1,\ x_3 = 0$$

- Solve by using inverse:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} 1/3 & 0 & 2/3 \\ 1/3 & 0 & -1/3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

(b) Systems with common coefficient matrix: To solve systems

$$A\mathbf{x} = \mathbf{b}_1, \qquad A\mathbf{x} = \mathbf{b}_2, \qquad \cdots, A\mathbf{x} = \mathbf{b}_k$$

we can use

- inverse: $\mathbf{x}_1 = A^{-1}\mathbf{b}_1, \qquad \mathbf{x}_2 = A^{-1}\mathbf{b}_2, \cdots, \mathbf{x}_k = A^{-1}\mathbf{b}_k$

- Gauss-Jordan elimination method

$$\left[A \,\vdots\, \mathbf{b}_1 \,\vdots\, \mathbf{b}_2 \,\vdots\, \cdots \,\vdots \mathbf{b}_k \right]$$