(1) **Diagonal matrix:** Square matrix with all entries off main diagonal equal 0

$$D_n = \begin{bmatrix} d_1 & & & & \\ & d_2 & & 0 & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & d_n \end{bmatrix} \implies D_n^k = \begin{bmatrix} d_1^k & & & & \\ & d_2^k & & 0 & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & d_n^k \end{bmatrix}$$

D_n is invertible iff $d_i \neq 0$ for all $i = 1, \cdots, n$. If D_n is invertible, then

$$D_n^{-1} = \begin{bmatrix} 1/d_1 & & & & \\ & 1/d_2 & & 0 & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & 1/d_n \end{bmatrix}$$

(2) **Symmetric matrix:** A is symmetric if $A = A^T$

Theorem:

(a) If A is symmetric, then A^T, kA are also symmetric.
(b) If A, B are symmetric matrices of same size, then $A + B$ and $A - B$ are symmetric.
(c) If A is symmetric and invertible, then A^{-1} is also symmetric.
(d) A^TA and AA^T are symmetric for any square matrix A.
(e) Diagonal matrice D_n is symmetric.

Proof:

(a) $(A^T)^T = A = A^T$. and $(kA)^T = kA^T = kA$
(b) $(A \pm B)^T = A^T \pm B^T = A \pm B$
(c) $(A^{-1})^T = (A^T)^{-1} = A^{-1}$
(d) $(AA^T)^T = (A^T)^T A^T = AA^T$. Likewise for A^TA.
(e) Diagonal matrice has properties $D_n^T = D_n$. Therefore it's symmetric.

$\star\star\star$ Note:

- Product of symmetric matrices does not necessarily symmetric. However, in the case that A and B are symmetric *and* $AB = BA$, then AB is symmetric.
  Proof: $(AB)^T = B^TA^T = BA = AB$.
- Symmetric matrix needs not be invertible. *(Can you provide an example?)*

(3) Examples:

Ex.1: What is the maximum number of distinct entries that a 3×3 symmetric matrix can have? How about $n \times n$?

Ex.2: Let A be an $n \times n$ symmetric matrix. Show that A^2 and $2A^2 - 3A + I$ are symmetric.

Ex.3: If $A = A^TA$, then A is symmetric and $A^2 = A$
Ex.4: True/False: For any square matrix A, $A + A^T$ is symmetric.
Ex.5: True/False: For any square matrix A, $A - A^T$ is symmetric.