1. Determinant by Cofactor Expansion

(1) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies \qquad det(A) = ad - bc.$

(2) Find determinant by cofactor:
- Def.: Minor entry of $a_{ij} = M_{ij}$ = determinant of submatrix that remained after deleting i^{th} row and j^{th} column.
- Cofactor $C_{ij} = (-1)^{i+j} M_{ij}$.
- Cofactor expansion along j^{th} column:

$$det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \cdots + a_{nj}C_{nj}$$

- Cofactor expansion along i^{th} row:

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots + a_{in}C_{in}$$

2. Evaluating Determinant by Row Reduction

Elementary row operations on A and their affects on $det(A)$:

Multiply a row/column by nonzero scalar k: $\implies k det(A)$
Interchange two rows/columns: $\implies -det(A)$
Multiply a row/column by nonzero scalar and add to other row/column: $\implies det(A)$

Find determinant by row reduction: For computation, we can reduce $n \times n$ matrix to a triangular matrix, whose determinant can be computed easily. We need to keep track of the operations so that we can relate this determinant to that of the original matrix.

3. Properties of Determinants

(1) *Basic properties:* Let A be an $n \times n$ matrix.
 (a) If A is a triangular matrix then $det(A)$ = product of main diagonal entries.
 (b) If A has a row/column of zeros, then $det(A) = 0$.
 (c) If A has two proportional rows or columns, then $det(A) = 0$.
 (d) $det(kA) = k^n det(A)$
 (e) In general, $det(A + B) \neq det(A) + det(B)$
 (f) $det(A^T) = det(A)$

(2) Lemma: If A is $n \times n$ matrix and E is $n \times n$ elementary matrix, then $det(EA) = det(E)det(A)$.

(3) A square matrix A is invertible if and only if $det(A) \neq 0$.
Proof:
Suppose that A is invertible, then

$$E_k \cdots E_2 E_1 A = I \quad \text{and} \quad det(E_k) \cdots det(E_2) det(E_1) det(A) = det(I)$$

But $det(I) = 1$ and $det(E_1), det(E_2), \cdots det(E_k)$ are not 0. Thus $det(A) \neq 0$.
Now suppose that $det(A) \neq 0$. Let R be the reduced row echelon form of A and $E_1, E_2 \cdots, E_k$

be elementary matrices that correspond to the elementary row operations that product R from A. Then,

$$R = E_k \cdots E_2 E_1 A$$
$$det(R) = det(E_k) \cdots det(E_2) det(E_1) det(A) \neq 0$$

Thus R cannot have a row of zero. Hence, $R = I$; that is, A is invertible.

(4) If A and B are square matrix of same size, then

$$det(AB) = det(A)det(B)$$

Proof:
Suppose A is not invertible, then AB is not invertible. Thus, $det(A) = 0$ and $det(AB) = 0$. Therefore, $det(AB) = det(A)det(B)$
Suppose that A is invertible. Then

$$E_k \cdots E_2 E_1 A = I \implies A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}.$$

Since invert of an elementary matrix is also an elementary matrix, we can rewrite A as $A = E_1 E_2 \cdots E_k$. Now,

$$\begin{aligned} AB &= E_1 E_2 \cdots E_k B \\ det(AB) &= det(E_1 E_2 \cdots E_k B) = det(E_1) det(E_2 \cdots E_k B) \quad \text{[by using lemma (2)]} \\ &= det(E_1) det(E_2) \cdots det(E_k) det(B) \\ &= det(E_1 E_2 \cdots E_k) det(B) = det(A) det(B) \end{aligned}$$

(5) If A is invertible, then $det(A^{-1}) = \frac{1}{det(A)}$

4. Theorem: Equivalent Statements

For any $n \times n$ matrix A, the following statements are equivalence:

(1) A is invertible.
(2) $A\mathbf{x} = \mathbf{0}$ has only trivial solution.
(3) The reduced row echelon from of A is I_n
(4) A is expressible as a product of elementary matrices.
(5) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
(6) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix B.
(7) $det(A) \neq 0$