1. VECTORS IN $\mathbb{R}^n$

Vectors are quantities measured by distance (magnitude) and direction.

- In $2-dim$ (or $2-space$) $\mathbb{R}^2$, we represent vector as ordered pair (x, y) or as matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. x and y are called components of the vector. In 3−space $\mathbb{R}^3$, vector is ordered triple (x, y, z) or $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.
- In n-space $\mathbb{R}^n$, we denote vector $\mathbf{u} = (u_1, \ldots, u_n)$ or $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}^T$.
  NOTE: Doesn't matter how we view $(u_1, \cdots, u_n)$ – as a point, a vector, or a matrix – the algebraic properties is still the same.
- Zero vector in $\mathbb{R}^n$: $\mathbf{0} = (0, \ldots, 0)$.
- Negative vector: -**u**= $(-u_1, \ldots, -u_n)$ has the same magnitude as **u**, but in opposite direction.
- Standard basis vectors in R^n: Vector of length 1 along coordinate axes.
  In $\mathbb{R}^2$: $\mathbf{i} = (1, 0)$, $\mathbf{j} = (0, 1)$
  In $\mathbb{R}^3$: $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$
  In $\mathbb{R}^n$: $\mathbf{e}_1 = (1, 0, 0, \cdots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \cdots, 0), \cdots, \mathbf{e}_n = (0, 0, \ldots, 0, 1)$.

2. VECTOR OPERATIONS

Vector operates the same way in $\mathbb{R}^n$ for all finite n. Even though it's hard to visualize in $n-$space for $n > 3$, we still can work with analytic or numerical properties in those space.
Let $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ be vectors in $\mathbb{R}^n$. Let c be a scalar.

- Equality: $\mathbf{u} = \mathbf{v}$ if $u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$.
- Addition: $\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n)$.
- Substraction: $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$.
- Scalar multiplication: $c\mathbf{u} = (cu_1, \ldots, cu_n)$.

3. PROPERTIES OF VECTOR OPERATIONS

Let $\mathbf{u} = (u_1, \ldots, u_n)$, $\mathbf{v} = (v_1, \ldots, v_n)$, and $\mathbf{w} = (w_1, \ldots, w_n)$ be vectors in $\mathbb{R}^n$, k and l be scalars.

Theorem:

(1) $\mathbf{u} + \mathbf{v}$ is a vector in $\mathbb{R}^n$. We say $\mathbb{R}^n$ is closed under vector addition.

(*a*) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(*b*) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

(*c*) Unique existence of **0** vector: $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(*d*) Unique existence of $-\mathbf{u}$ vector: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;

(2) $c\mathbf{u}$ is a vector in $\mathbb{R}^n$. We say $\mathbb{R}^n$ is closed under scalar multiplication.

(*e*) $k(l\mathbf{u}) = (kl)\mathbf{u}$

(*f*) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(*g*) $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$

(*h*) $1\mathbf{u} = \mathbf{u}$

Proof: (1) and (2) are from definition of vector operations.
(*a*) $\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n) = (v_1 + u_1, \ldots, v_n + u_n) = \mathbf{v} + \mathbf{u}$.

$(d)\ \ \mathbf{u}+(-\mathbf{u})=(u_1,\dots,u_n)+(-u_1,\dots,-u_n)=(u_1-u_1,\dots,u_n-u_n)=(0,0,\dots,0)=\mathbf{0}$
$(g)\ (k+l)\mathbf{u}=(k+l)(u_1,\dots,u_n)=((k+l)u_1,\dots,(k+l)u_n)=(ku_1+lu_1,\dots,ku_n+lu_n)$

$$=(ku_1,\dots,ku_n)+(lu_1,\dots,lu_n)=k(u_1,\dots,u_n)+l(u_1,\dots,u_n)=k\mathbf{u}+l\mathbf{u}$$

4. Dot Product

Norm: (or Length) of a vector is defined as

$$\|\mathbf{u}\|=\sqrt{u_1^2+u_2^2+\cdots+u_n^2}$$

A *unit vector* is a vector whose length is 1. To normalize a vector $\mathbf{x}$ means to turn $\mathbf{x}$ into a unit vector by dividing by its norm:

$$\mathbf{u}=\frac{1}{\|\mathbf{x}\|}\mathbf{x}. \quad \text{Note that } \mathbf{u} \text{ is a unit vector.}$$

Dot product (Euclidean inner product): Standard dot product is a scalar.

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+\cdots+u_nv_n=\begin{cases}\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta, & \text{if } \mathbf{u}\neq\mathbf{0} \text{ and } \mathbf{v}\neq\mathbf{0}\\ 0, & \text{if } \mathbf{u}=\mathbf{0} \text{ or } \mathbf{v}=\mathbf{0}\end{cases}$$

Theorem: $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$

Angle between two vectors: $\cos\theta=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$, where $0\le\theta\le\pi$.

Properties of dot product:

$(a)\quad \mathbf{u}.\mathbf{v}=\mathbf{v}.\mathbf{u}$
$(b)\quad (\mathbf{u}+\mathbf{v}).\mathbf{w}=\mathbf{u}.\mathbf{w}+\mathbf{v}.\mathbf{w}$
$(c)\quad (k\mathbf{u}).\mathbf{v}=\mathbf{u}.(k\mathbf{v})=k(\mathbf{u}.\mathbf{v})$
$(d)\quad \mathbf{v}.\mathbf{v}\ge 0$. Further $\mathbf{v}.\mathbf{v}=0$ iff $\mathbf{v}=\mathbf{0}$

Proof: similar technique as the proofs above.

$(a)\ \mathbf{u}.\mathbf{v}=u_1v_1+\cdots+u_nv_n=v_1u_1+\cdots+v_nu_n=\mathbf{v}.\mathbf{u}$
$(c)\ (k\mathbf{u}).\mathbf{v}=(ku_1,\dots,ku_n).(v_1,\dots,v_n)$
$\quad =ku_1v_1+\cdots+ku_nv_n=k(u_1v_1+\cdots+u_nv_n)=k(\mathbf{u}.\mathbf{v})$

Cauchy-Schwarz inequality: $|\mathbf{u}.\mathbf{v}|\le\|\mathbf{u}\|\|\mathbf{v}\|$

Triangle inequality: $\|\mathbf{u}+\mathbf{v}\|\le\|\mathbf{u}\|+\|\mathbf{v}\|$

Orthogonality: Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal iff $\mathbf{u}.\mathbf{v}=\mathbf{0}$.

Theorem: If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vector in $\mathbb{R}^n$, then

$$\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2$$

Proof: $\|\mathbf{u}+\mathbf{v}\|^2=(\mathbf{u}+\mathbf{v}).(\mathbf{u}+\mathbf{v})=\mathbf{u}.\mathbf{u}+2\mathbf{u}.\mathbf{v}+\mathbf{v}.\mathbf{v}=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2$.

Orthogonal Projection: of $\mathbf{u}$ on $\mathbf{v}$

$$\text{proj}_{\mathbf{v}}\mathbf{u}=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}, \qquad \text{Note: } \|\text{proj}_{\mathbf{v}}\mathbf{u}\|=\frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{v}\|}$$

Vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ is $\mathbf{u}-\text{proj}_{\mathbf{v}}\mathbf{u}$

Distance: between two vectors (or two points) is defined as

$$\|\mathbf{u}-\mathbf{v}\|=\sqrt{(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})}=\sqrt{(u_1-v_1)^2+\cdots+(u_n-v_n)^2}$$