In this section, we only consider vectors in 3−space. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

Definition: the cross product $\mathbf{u} \times \mathbf{v}$ is a vector defined by

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \ -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \ \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

The orientation of $\mathbf{u} \times \mathbf{v}$ is determined by the right hand rule.

Angle: between two vectors

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta, \quad \text{where } 0 \le \theta \le \pi$$

Parallel property: Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are <u>parallel</u> iff $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Properties of cross product:

- $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. In fact, it is orthogonal to the plane formed by $\mathbf{u}$ and $\mathbf{v}$.
$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0, \qquad \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$$
- Lagrange's identity $\quad \|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$
- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.
- $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$.
- $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$.
- $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$.
- $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.
- $\|\mathbf{u} \times \mathbf{v}\|$ is equal to the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.
- $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ represents the volume of parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.