Definitions:

- <u>Transformation from $\mathbb{R}^n \to \mathbb{R}^m$</u>:

$$\begin{aligned} T : \mathbb{R}^n &\to \mathbb{R}^m \\ T\underbrace{(x_1, \dots, x_n)}_{\text{domain}} &= \underbrace{(w_1, \dots, w_m)}_{\text{co-domain}} \end{aligned}$$

  T is called a *transformation* from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- If $n = m$, then $T : \mathbb{R}^n \to \mathbb{R}^n$ is called an *operator* on $\mathbb{R}^n$.
- $T : \mathbb{R}^n \to \mathbb{R}^m$; T is linear iff for all vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ and every scalar c,

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(c\mathbf{u}) = cT(\mathbf{u}).$$

Theorem: For every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, there is a corresponding $m \times n$ matrix A that represents T.

$$T(\mathbf{x}) = A\mathbf{x} = \mathbf{w}.$$

A is called standard matrix for T and T is called multiplication by A.

Basic theorems: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, $\mathbf{u}_1, \cdots, \mathbf{u}_k$ be vectors in $\mathbb{R}^n$, and $c_1, \cdots, c_k$ be scalars.

- $T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) = c_1T(\mathbf{u}_1) + c_2T(\mathbf{u}_2) + \cdots + c_kT(\mathbf{u}_k)$.
- $T(\mathbf{0}_{R^n}) = \mathbf{0}_{R^m}$.
  Corollary: If $T(\mathbf{0}_{R^n}) \neq \mathbf{0}_{R^m}$, then T is a nonlinear transformation.
- $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$

Some Basic Linear Operators: on $\mathbb{R}^2$ and $\mathbb{R}^3$

(1) *Zero transformation*: $T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$.
Identity transformation: $T_I(\mathbf{x}) = I_n\mathbf{x} = \mathbf{x}$.

(2) *Reflection Operator*: Map each vector into its symmetric image about a line or a plane.

<u>Ex.</u>: : reflect about y-axis $T : \mathbb{R}^2 \to \mathbb{R}^2$ $(x, y) \mapsto (-x, y)$ $T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$	<u>Ex.</u>: reflect about xz-plane $T : \mathbb{R}^3 \to \mathbb{R}^3$ $(x, y, z) \mapsto (x, -y, z)$ $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	<u>Ex.</u>: reflect about $x = y$-line $T : \mathbb{R}^2 \to \mathbb{R}^2$ $(x, y) \mapsto (y, x)$ $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(3) *Orthogonal projection operator*: Map each vector to its orthogonal projection
<u>Ex.</u>: $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$(x, y) \mapsto (0, y) : \text{ project on } y\text{-axis, } \mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(x, y) \mapsto (x, 0) : \text{ project on } x\text{-axis}$$

<u>Ex.</u>: $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $(x, y, z) \mapsto (x, y, 0)$: project on xy-plane.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(4) *Rotation Operator*: Rotate counter-clockwise each vector by some fixed angle θ in $\mathbb{R}^2$. Let $x = r\cos\phi,\ y = r\sin\phi$.

$$(r\cos\phi, r\sin\phi) \mapsto (r\cos(\phi+\theta), r\sin(\phi+\theta))$$

$$\begin{aligned} w_1 &= r\cos\phi\cos\theta - r\sin\phi\sin\theta = x\cos\theta - y\sin\theta \\ w_2 &= r\sin\phi\cos\theta - r\cos\phi\sin\theta = y\cos\theta + x\sin\theta \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(5) *Dilation/Contraction Operators*:

$$\begin{aligned} T : \mathbb{R}^3 &\to \mathbb{R}^3 \\ \mathbf{x} &\mapsto k\mathbf{x}, \qquad k \text{ is a positive scalar.} \end{aligned}$$

Contraction $0 \le k \le 1$
Dilation: $k \ge 1$. (Stretch)

(6) *Composition of linear transformation*:

$$(T_2 \circ T_1)(\mathbf{x}) = T_2\left(T_1(\mathbf{x})\right) = A_{T_2}A_{T_1}$$