1. **Vector space axioms:** Let V be a set on which 2 operations: vector addition $\oplus$ and scalar multiplication $\odot$ are defined. V is called a *vector space* (over real number) if the following axioms are satisfied.
   - Closure Axioms:
     - **1:** Sum $\mathbf{u} \oplus \mathbf{v}$ exists and is element of V. (Closure under addition).
     - **2:** $c \odot \mathbf{u}$ is an element of V. (Closure under scalar multiplication).
   - Addition Axioms
     - **3:** $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$ (Commutative).
     - **4:** $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$ (Associative)
     - **5:** There exists an element of V, called the zero vector, denoted $\mathbf{0}$, such that $\mathbf{u} \oplus \mathbf{0} = \mathbf{u}$
     - **6:** For every element $\mathbf{u}$ of V, there exists an element called the negative of $\mathbf{u}$; denoted $-\mathbf{u}$, such that $\mathbf{u} \oplus (-\mathbf{u}) = \mathbf{0}$.
   - Scalar Multiplication Axioms
     - **7:** $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot \mathbf{u} \oplus c \odot \mathbf{v}$
     - **8:** $(c + d) \odot \mathbf{u} = c \odot \mathbf{u} \oplus d \odot \mathbf{u}$
     - **9:** $(cd) \odot \mathbf{u} = c \odot (d \odot \mathbf{u})$
     - **10:** $1 \odot \mathbf{u} = \mathbf{u}$

2. **Common vector spaces:** under standard addition and scalar multiplication.
   - $\mathbb{R}^n$: set of n-tuples.
   - $M_{m \times n}$: set of $m \times n$ matrices.
   - P_n: set of polynomials degree less than or equal n.
   - C^1: set of continuously differentiable functions.

3. **Examples:**

   (1) Prove that P_n with standard addition and scalar multiplication is a vector space.
   Ans.: verify it satisfies all 10 axioms.

   (2) Let $V = \{(x, y) \in \mathbb{R}^2 : y = 2x\}$ with standard addition and scalar multiplication. Is V a vector space?
   Let $\mathbf{u} = (u, 2u)$ and $\mathbf{v} = (v, 2v)$ be vectors in V.
   - $\mathbf{u} + \mathbf{v} = (u + v, 2u + 2v) = (u + v, 2(u + v))$: passes axiom 1.
   - $c\mathbf{u} = (cu, 2cu)$: passes axiom 2.
   - $\mathbf{u} + \mathbf{v} = (u + v, 2(u + v))$ and $\mathbf{v} + \mathbf{u} = (v + u, 2(v + u)) = (u + v, 2(u + v))$; thus $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$: passes axiom 3.
   - $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (u, 2u) + (v + w, 2(v + w)) = (u + v + w, 2(u + v + w))$.
     $(\mathbf{u} + \mathbf{u}) + \mathbf{w} = (u + v, 2(u + v)) + (w, 2w) = (u + v + w, 2(u + v + w))$.
     Passes axiom 4.
   - Note that $\mathbf{0} = (0, 0)$ is in V. In addition, $\mathbf{u} + \mathbf{0} = (u, 2u) + (0, 0) = \mathbf{u}$. So it passes axiom 5.
   - Since $-\mathbf{u} = (-u, -2u)$ is in V and $\mathbf{u} + (-\mathbf{u}) = (u, 2u) + (-u, -2u) = (0, 0) = \mathbf{0}$, it passes axiom 6.
   - $c(\mathbf{u} + \mathbf{v}) = c(u + v, 2(u + v)) = (c(u + v),\ 2c(u + v)) = (cu + cv,\ 2cu + 2cv) = (cu, 2cu) + (cv, 2cv) = c\mathbf{u} + c\mathbf{v}$: passes axiom 7.
   - $(c + d)\mathbf{u} = ((c + d)u, (c + d)2u) = (cu + du, 2cu + 2du) = (cu, 2cu) + (du, 2du) = c\mathbf{u} + d\mathbf{u}$: passes axiom 8.
   - $(cd)\mathbf{u} = (cdu, 2cdu)$ and $c(d\mathbf{u}) = c(du, 2du) = (cdu, 2cdu)$: passes axiom 9.
   - $1 \cdot \mathbf{u} = (u, 2u) = \mathbf{u}$: passes axiom 10.

   Therefore, V is a vector space (over real number).

(3) Let $V = \mathbb{R}$. For any x and y in V, defined

$$x \oplus y = xy, \qquad c \odot x = x^c.$$

Does V form a vector space? What if V is set of positive real numbers $x > 0$?

When $V = \mathbb{R}$, it fails axiom 2. For instant $\frac{1}{2} \odot (-4) = 2i \notin \mathbb{R}$.
When $V = \{x \in \mathbb{R} : x > 0\}$: we can easily show that it passes axioms 1 - 4 and 7 - 10.

- Axiom 5: 1 is in V and $x \oplus 1 = x \cdot 1 = x$; therefore, the zero vector in this case is 1: $\mathbf{0} = 1$.
- Axiom 6: Since $x \oplus \frac{1}{x} = x \cdot \frac{1}{x} = 1 = \mathbf{0}$. This means the negative of x is $\frac{1}{x}$.

(4) Let V be set of even integers and define

$$x \oplus y = 3x + 3y, \qquad c \odot x = 2cx.$$

(a) Is V a vector space over real numbers?

No. It passes axiom 1 but fails axiom 2: $\frac{1}{4} \odot 2 = 2 \cdot \frac{1}{4} \cdot 2 = 1$, not an even integer.

(b) Is V a vector space over integers; i.e, scalar c is an integer?
It is easy to show that it passes axioms 1 - 6.

- $c \odot (x \oplus y) = c \odot (3x + 3y) = 2c(3x + 3y) = 6cx + 6cy$ and $(c \odot x) \oplus (c \odot y) = (2cx) \oplus (2cy) = 3(2cx) + 3(2cy) = 6cx + 6cy$. It passes axiom 7.
- $(c + d) \odot x = 2(c + d)x$ while $c \odot x \oplus d \odot x = (2cx) \oplus (2dx) = 6cx + 6dy$: fails axiom 8.
- $(cd) \odot x = 2cdx$ while $c \odot (d \odot x) = c \odot (2dx) = 2c(2dx) = 4cdx$: fails axiom 9.
- $1 \odot x = 2(1)x = 2x$: fails axiom 10.