1. Subspace: Let V be a vector space and $\mathbf{W}$ is a non-empty subset of V. $\mathbf{W}$ is a subspace of V if it is closed under addition and scalar multiplication.
$\star$ Every non-zero vector space V has 2 trivial subspaces: itself and $\mathbf{0}$ space. Subspace always contains zero vector.

2. Linearly independent and span: Let V be a vector space and $W = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$ be a set of vectors in V.

(1) *Def.*: A vector $\mathbf{w}$ is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ if it can be expressed in form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

Ex.: Every vector $\mathbf{v} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ is a linear combination of $(1,0,0), (0,1,0), (0,0,1)$.

(2) *Def.*: The set of all linear combinations of W is called span of W. We denote:

$$span(W) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \cdots c_r\mathbf{v}_r\}$$

(3) *Def.*: W is linearly independent if and only if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0} \implies c_1 = \cdots = c_n = 0$$

If there is other solution, then W is linear dependent.

3. Important theorems:

(1) $span(W)$ is a subspace of V.
(2) Set S with two or more vectors is linearly dependent iff at least one of the vectors in S is expressible as linear combination of the other vectors in S.
(3) A finite set of vectors that contains the zero vector is linearly dependent.
(4) A set with exactly two vectors is linearly dependent iff one is a scalar multiple of the other.
(5) Let $W = \{v_1, \ldots, v_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then W is linearly dependent (This means linearly independent set in $\mathbb{R}^n$ can contain at *most* n vectors.)