I. Basis:

(1) Definition:
If V is any vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a set of vectors in V, then S is called a **basis** for V if
  (a) S is linearly independent.
  (b) S spans V.

(2) Standard basis for $\mathbb{R}^n$:
Let $\mathbf{e}_1 = (1, 0, \cdots, 0)$, $\mathbf{e}_2 = (0, 1, \cdots, 0)$, $\cdots$, $\mathbf{e}_n = (0, \cdots, 0, 1)$.

$$\text{Then } S = \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$$

is called standard basis for $\mathbb{R}^n$.

(3) Theorem:
If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

in *exactly one* way.

(4) Definition:
If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for vector space V, and

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

Then $(k_1, \cdots, k_n)$ is called *coordinates* of $\mathbf{v}$ relative to the basis S.

(5) Example:

  (a) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is standard basis for $\mathbb{R}^3$.
  (b) $S = \{(1, 2), (-1, 1)\}$ is a basis for $\mathbb{R}^2$. Find the coordinate of $\mathbf{v} = (2, -2)$ with respect to S.
  We must find c_1, c_2 such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$

$$\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Longrightarrow \begin{matrix} c_1 = 0 \\ c_2 = -2 \end{matrix}$$

  (c) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is basis for $M_{2,2}$.
  (d) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is basis for 2×2 symmetric matrices.
  (e) $\{1, x, x^2, \cdots, x^n\}$ is a basis for P_n.

II. Dimension:

(1) Definition:
The **dimension** of a finite-dimensional vector space V, denoted $dim(V)$, is defined to be the number of vectors in a basis of V.
Zero vector space has dimension 0.

(2) Some important dimensions:

$$dim(\mathbb{R}^n) = n$$
$$dim(P_n) = n + 1$$
$$dim(M_{mn}) = mn$$

(3) Theorem:
If V is an n-dimensional vector space, and if S is a set in V with exactly n vectors, then S is a basis for V if either S spans V or S is linearly independent.

(4) Theorem:
Let S be a set of vectors in a finite-dimensional vector space V.
  (a) If S spans V but is not a basis for V, then S can be reduce to a basis for V by removing appropriate vectors from S.
  (b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

(5) Theorem:
If W is a subspace of finite-dimensional vector space V, then $dim(W) \leq dim(V)$. If $dim(W) = dim(V)$, then $W = V$.