§5.5-5.6: ROW SPACES, COLUMN SPACES, AND NULL SPACES

Def.: Let A be an $m \times n$ matrix.

- Row space: denoted $row(A)$, is a subspace of $\mathbb{R}^n$ spanned by row vectors of A.
- Column space: denoted $col(A)$, is a subspace of $\mathbb{R}^m$ spanned by column vectors of A.
- Null space: The solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is called a **nullspace** of A; denoted $null(A)$. $null(A)$ is a subspace of $\mathbb{R}^n$.

Finding Basis: Let $R = rref(A)$. Then

- Non-zero row vectors of R form a basis for $row(A)$.
- Column vectors of A that corresponding to column of R containing leading 1's form a basis for $col(A)$.
- Solve the system $A\mathbf{x} = \mathbf{0}$ for null space.

Rank and Nullity:

(1) Theorem: If A is $m \times n$ matrix, then $dim(row(A)) = dim(col(A))$.

(2) Def.:
- Dimension of $row(A)$ is called *rank* of A, denoted $rank(A)$.
- Dimension of nullspace of A is called *nullity* of A, denoted $nullity(A)$.

(3) Dimension Theorem: A is $m \times n$ matrix.

$$rank(A) + nullity(A) = n$$

Remark: $rank(A) = rank(A^T)$ *and* $rank(A) \leq min(m, n)$

Examples:

Ex 1:

$$\text{Let } A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \text{ Then } R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } rref(A^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Basis for row space $= \{(1,0,0), (0,1,0), (0,0,1)\}$.
Basis for column space $= \{(2,1,0), (0,1,1), (-1,2,1)\}$.
Since $\mathbf{x} = 0$ is solution of $A\mathbf{x} = 0$, basis for nullspace is $\{(0,0,0)\}$.
$Rank(A) = 3$, $nullity(A) = 0$. Verifying dimension theorem: $3 + 0 = 3$.

Ex 2:

$$\text{Let } A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 3 & 1 & -5 & 0 \\ 1 & 2 & 0 & -5 \end{bmatrix} \text{ Then } R = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } rref(A^T) = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for row space $= \{(1,0,-2,1), (0,1,1,-3)\}$.
Basis for column space $= \{(1,3,1), (0,1,2)\}$.

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Solving $A\mathbf{x} = \mathbf{0}$, we get

$$\begin{aligned} x_1 &= 2t - s \\ x_2 &= -t + 3s \\ x_3 &= t \\ x_4 &= s \end{aligned} \implies \mathbf{x} = t(2,-1,1,0) + s(-1,3,0,1)$$

So basis for nullspace $= \{(2,-1,1,0), (-1,3,0,1)\}$.
$Rank(A) = 2$, $nullity(A) = 2$, and verifying dimension theorem: $2 + 2 = 4$.
Example 3:

$$\text{Let } A = \begin{bmatrix} 3 & 1 & 0 & 2 \\ 2 & -1 & 5 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix} \text{ Then } R = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Basis for row space $= \{(1,0,0,3), (0,1,0,-7), (0,0,1,-2)\}$.
Basis for column space $= \{(3,2,1), (1,-1,0), (0,5,2)\}$.
Solving $A\mathbf{x} = \mathbf{0}$, we get

$$\begin{aligned} x_1 &= -3t \\ x_2 &= 7t \\ x_3 &= 2t \\ x_4 &= t \end{aligned} \implies \mathbf{x} = t(-3,7,2,1)$$

Therefore, basis for nullspace $= \{(-3,7,2,1)\}$.
Ex 4:

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & -5 \\ 2 & 2 & 1 \\ -1 & 0 & -2 \end{bmatrix} \text{ Then } R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } rref(A^T) = \begin{bmatrix} 1 & 0 & 0 & 19/3 \\ 0 & 1 & 0 & 8/3 \\ 0 & 0 & 1 & -23/3 \end{bmatrix}$$

Basis for row space

Basis for column space

Basis for null space

$$\underline{\text{Ex.5}}\text{: } A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ -1 & 3 & -2 & 0 & -3 \\ 0 & 1 & 3 & 1 & 0 \\ 1 & -1 & 8 & 0 & 3 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 11 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for row space

Basis for column space

Basis for null space

Number of Solutions: of a system of linear equations:
Let $A\mathbf{x} = \mathbf{b}$ be a system of linear equations in n variables.
(1) If $rank(A) = rank([A : \mathbf{b}]) = n$, then the system has unique solution.
(2) If $rank(A) = rank([A : \mathbf{b}]) < n$, then the system has infinite number of solutions.
(3) If $rank(A) \neq rank([A : \mathbf{b}])$, then the system has no solution.

Proof: Let the system of linear equations be

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Let $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$ be column vectors of coefficient matrix. Then the system can be expressed as

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b}.$$

(1) Since $rank(A) = rank([A : \mathbf{b}])$, $\mathbf{b}$ must be linearly dependent on $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$. Furthermore, since $rank(A) = n$, $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$ are linearly independent and thus form a basis for $col([A : \mathbf{b}])$. Thus, $\mathbf{b}$ can be expressed *uniquely* as linear combination of $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$; i.e., the system has unique solution.

(2) Similar to above, $rank(A) = rank([A : \mathbf{b}])$ implies $\mathbf{b}$ must be linearly dependent on $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$. However, since $rank(A) < n$, $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$ are linearly dependent. Thus $\mathbf{b}$ can be expressed in more than one way as linear combination of $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$. Hence, the system has infinitely many solutions.

(3) Since $rank(A) \neq rank([A : \mathbf{b}])$, $\mathbf{b}$ must be linearly independent on $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$. Thus, the system has no solution.
Note: Indeed, $rank(A) < rank([A : \mathbf{b}])$ in this case.

Remark: The last theorem states that the system $A\mathbf{x} = \mathbf{b}$ is consistent iff $rank(A) = rank([A : \mathbf{b}])$. This is equivalent to saying that the system $A\mathbf{x} = \mathbf{b}$ is consistent iff $\mathbf{b}$ is in the column space of A.
Ex 6:

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 3 & 1 & -5 & 0 \\ 1 & 2 & 0 & -5 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 5 \\ 8 \\ -9 \end{bmatrix} \text{ Then } rref([A : \mathbf{b}]) = \begin{bmatrix} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note $rank(A) = rank([A : \mathbf{b}]) < n$, and we can see that the system has infinitely many solutions.

Theorem: Let $\mathbf{x}$ be solution to nonhomogenous system

$$A\mathbf{x} = \mathbf{b}, \ \mathbf{b} \neq \mathbf{0}. \qquad (\star)$$

Then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_p$ is a particular solution to the equation $(\star)$ and $\mathbf{x}_h$ is solution to corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

Ex.4: Let $A = \begin{bmatrix} 1 & 2 & 0 & -3 & 1 & 0 \\ 1 & 2 & 1 & -3 & 1 & 2 \\ 1 & 2 & 0 & -3 & 2 & 1 \\ 3 & 6 & 1 & -9 & 4 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 9 \end{bmatrix}$. Then

$$A\mathbf{x} = \mathbf{b} \implies \mathbf{x} = \begin{bmatrix} r + 3s - 2t \\ t \\ 1 - 2r \\ s \\ 2 - r \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + r\begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, $\mathbf{x}_p = (0, 0, 1, 0, 2, 0)$.
$null(A) = \{(1, 0, -2, 0, -1, 1),\ (3, 0, 0, 1, 0, 0),\ (-2, 1, 0, 0, 0, 0)\}$, nullity$(A) = 3$.