Motivation:
In section 4.1, we extended the concept of length and distance from $\mathbb{R}^2$ to $\mathbb{R}^n$ by using dot product on $\mathbb{R}^n$. The dot product, called Euclidean inner product, is only one of several inner product that can be defined on $\mathbb{R}^n$. We will extend the concept one step further–inner product for general vector spaces. To distinguish between standard inner product and other possible inner product, we use notation:

$\mathbf{u}.\mathbf{v} =$ Euclidean inner (dot) product
$\langle \mathbf{u}, \mathbf{v} \rangle =$ general inner product for vector space V.

I **Definition**:

Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ and satisfies the following axioms:

(1) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ (symmetry)
(2) $\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ (additivity)
(3) $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$ (homogeneity)
(4) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ iff $\mathbf{v} = \mathbf{0}$. (positivity)

Remark: A vector space with an inner product is called inner product space.

<u>Ex 1</u>: Euclidean inner product for $\mathbb{R}^n$
Let $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}.\mathbf{v} = u_1v_1 + u_2v_2 + \cdots u_nv_n$. Verify the 4 axioms above.

<u>Ex 2</u>: A different inner product for $\mathbb{R}^3$: weight Euclidean inner product
Let $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + u_2v_2 + 3u_3v_3$. Verify the axioms:

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v} \rangle &= 2u_1v_1 + u_2v_2 + 3u_3v_3 = 2v_1u_1 + v_2u_2 + 3v_3u_3 = \langle \mathbf{v}, \mathbf{u} \rangle \\
\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle &= 2(u_1+v_1)w_1 + (u_2+v_2)w_2 + 3(u_3+v_3)w_3 \\
&= 2u_1w_1 + 2v_1w_1 + u_2w_2 + v_2w_2 + 3u_3w_3 + 3v_3w_3 = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle \\
\langle c\mathbf{u}, \mathbf{v} \rangle &= 2cu_1v_1 + cu_2v_2 + 3cu_3v_3 = c(2u_1v_1 + u_2v_2 + 3u_3v_3) = c\langle \mathbf{u}, \mathbf{v} \rangle \\
\langle \mathbf{v}, \mathbf{v} \rangle &= 2v_1^2 + v_2^2 + 3v_3^2 \geq 0 \text{ and } \langle \mathbf{v}, \mathbf{v} \rangle = 0 \text{ iff } v_1 = v_2 = v_3 = 0; i.e., \mathbf{v} = \mathbf{0}.
\end{aligned}$$

<u>Ex 3</u>: Inner product on P_n
Let $\mathbf{p} = a_0 + a_1x + \cdots + a_nx_n$ and $\mathbf{q} = b_0 + b_1x + \cdots + b_nx_n$.
Define $\langle \mathbf{p}, \mathbf{q} \rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n$

<u>Ex 4</u>: Inner product on M_{mn}
Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ Define $\langle A,\ B \rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$.

<u>Ex 5</u>: Let V be vector space of differentiable functions. For $p(x)$ and $q(x)$ in V, define

$$\langle p(x),\ q(x) \rangle = \frac{d}{dx}[p(x)q(x)]\Big|_{x=0}$$

Determine if $\langle p(x),\ q(x) \rangle$ is an inner product on V.

II Some properties of inner product: Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in inner product space V, and let c be any real scalar.

$$(1)\ \langle \mathbf{0}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{0}\rangle = 0$$
$$(2)\ \langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle$$
$$(3)\ \langle \mathbf{u}, c\mathbf{v}\rangle = c\langle \mathbf{u}, \mathbf{v}\rangle$$

Proof:

$$(1)\ \langle \mathbf{0}, \mathbf{v}\rangle = \langle 0.\mathbf{v}, \mathbf{v}\rangle = 0\langle \mathbf{v}, \mathbf{v}\rangle = 0$$
$$(2)\ \langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{v}+\mathbf{w}, \mathbf{u}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{w}, \mathbf{u}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle$$
$$(3)\ \langle \mathbf{u}, c\mathbf{v}\rangle = \langle c\mathbf{v}, \mathbf{u}\rangle = c\langle \mathbf{v}, \mathbf{u}\rangle = c\langle \mathbf{u}, \mathbf{v}\rangle$$

III Definition of norm, distance, and angle:
Let $\mathbf{u}, \mathbf{v}$ be vectors in an inner product space V.

(1) Norm of $\mathbf{u}$ is $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle}$.
(2) Distance between $\mathbf{u}$ and $\mathbf{v}$ is $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.
(3) Angle between two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} 0 \le \theta \le \pi$$

(4) $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\langle \mathbf{u}, \mathbf{v}\rangle = 0$$

Remark:

- If $\|\mathbf{v}\| = 1$, then $\mathbf{v}$ is called unit vector.
- If $\mathbf{v}$ is any non-zero vector, then $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is unit vector and is called unit vector in direction of $\mathbf{v}$.

Ex 6: Let $\mathbf{p} = 1 - 2x^2$ and $\mathbf{q} = 4 - 2x + x_2$ be polynomials in P_2 with inner product as defined in example above.

$$\langle \mathbf{p}, \mathbf{q}\rangle = 4.1 + 0.(-2) + (-2)(-1) = 2$$
$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p}\rangle} = \sqrt{5}$$
$$\|\mathbf{q}\| = \sqrt{\langle \mathbf{q}, \mathbf{q}\rangle} = \sqrt{21}$$
$$d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = \|-3 + 2x - 3x^2\| = \sqrt{22}$$
$$\cos\theta = \frac{\langle \mathbf{p}, \mathbf{q}\rangle}{\|\mathbf{p}\|\|\mathbf{q}\|} = \frac{2}{\sqrt{5}\sqrt{21}} \Longrightarrow \theta = \cos^{-1}\left(\frac{2}{\sqrt{105}}\right).$$

Ex 7: Let A be an invertible $n \times n$ matrix. Then

$$\langle \mathbf{u}, \mathbf{v}\rangle = (A\mathbf{u}) \cdot \mathbf{v}$$

is an inner product on $\mathbb{R}^n$ generated by A.
Suppose that $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Compute the cosine of $\mathbf{u}$ and $\mathbf{v}$ where $\mathbf{u} = (-1, 0)$ and $\mathbf{v} = (2, 1)$.

Answer: see lecture notes.