(1) **Cauchy-Schwarz Inequality:**

$$| < \mathbf{u}, \mathbf{v} > | \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

(2) **Properties of length and distance in inner product space:**

Length	*Distance*
(a) $\|\mathbf{u}\| \geq 0$	(a) $d(\mathbf{u}, \mathbf{v}) \geq 0$
(b) $\|\mathbf{u}\| = 0$ iff $\mathbf{u} = \mathbf{0}$	(b) $d(\mathbf{u}, \mathbf{v}) = 0$ iff $\mathbf{u} = \mathbf{v}$
(c) $\|k\mathbf{u}\| = |k|\|\mathbf{u}\|$	(c) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
(d) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$	(d) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$

The last inequality (d) is called triangle inequality.
Proof are pretty much similar; just be careful that we deal with general inner product now.

(3) **Def.**
Let W be a subspace of an inner product space V. A vector $\mathbf{u}$ in V is said to be orthogonal to W if it is orthogonal to every vector in W, and the set of all vectors in V that are orthogonal to W is called the orthogonal complement of W, denote $W^{\perp}$.

(4) **Theorem:** If W be a subspace of finite dimensional inner product space V, then:
(a) $W^{\perp}$ is a subspace of V.
(b) $W \cap W^{\perp} = \mathbf{0}$.
(c) $(W^{\perp})^{\perp} = W$.

(5) **Theorem:** If A is an $m \times n$ matrix, then
(a) The nullspace of A and the row space of A are orthogonal complement in $\mathbb{R}^n$ with respect to Euclidean inner product.
(b) The nullspace of A^T and the column space of A are orthogonal complement in $\mathbb{R}^m$ with respect to Euclidean inner product.

Ex.:

$$\text{Let } A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 3 & 1 & -5 & 0 \\ 1 & 2 & 0 & -5 \end{bmatrix} \text{ Then } R = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for row space $= \{(1, 0, -2, 1), (0, 1, 1, -3)\}$.
Basis for col space $= \{(1, 3, 1), (0, 1, 2)\}$.
Solving $A\mathbf{x} = \mathbf{0}$, we get

$$x_1 = 2t - s,\ x_2 = -t + 3s,\ x_3 = t,\ x_4 = s$$

So basis for nullspace $= \{(2, -1, 1, 0), (-1, 3, 0, 1)\}$.
We can easily verify the theorem above.

Remark: $\{\mathbf{0}\}$ and V are orthogonal complement. This means the following statements are equivalent: Let A be an $n \times n$ matrix

- $A\mathbf{x} = \mathbf{0}$ has only trivial solution.
- Orthogonal complement of nullspace is all of $\mathbb{R}^n$.
- $row(A)$ is all $\mathbb{R}^n$.