(1) **Definition:**
A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in an inner product space is called orthogonal set if all pairs of distinct vectors in the set are orthogonal.

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \text{ for } i \neq j.$$

An orthogonal set in which each vector has norm 1 is called orthonormal.

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \text{ for } i \neq j \text{ and } \|\mathbf{v}_i\| = 1,\ i = 1, 2, \ldots, n.$$

Ex: $S = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}), (\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})\}$ is orthonormal.

(2) **Theorem:** An orthogonal set of nonzero vectors in an inner product space is linearly independent.
Proof:
Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthogonal set of nonzero vectors in an inner product space V. Consider $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \cdots + c_n\mathbf{v}_n = \mathbf{0}$. Then,

$$\langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n, \mathbf{v}_i \rangle = \langle \mathbf{0}, \mathbf{v}_i \rangle, \quad i = 1, \cdots, n$$
$$c_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2\langle \mathbf{v}_2, \mathbf{v}_i \rangle + \cdots + c_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle + \cdots + c_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle = 0.$$

By orthogonality, we get $c_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 0$. In addition, $\mathbf{v}_i \neq \mathbf{0}$ implies $\langle \mathbf{v}_i, \mathbf{v}_i \rangle > 0$. Thus, $c_i = 0$ and S is linearly independent.

Cor: If V is an inner product space of $dim(n)$, then any orthogonal set of n vectors is a basis of V.

(3) **Theorem**: Coordinate relative to an orthonormal basis:
If $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthonormal basis for the inner product space V, then the coordinate c_i of a vector $\mathbf{w}$ with respect to B is

$$c_i = \langle \mathbf{w}, \mathbf{v}_i \rangle \text{ and } \mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \cdots + c_n\mathbf{v}_n.$$

Proof: Since B is a basis for V, for any vector $\mathbf{w}$, there exists $c_1, \ldots, c_n$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \cdots + c_n\mathbf{v}_n.$$

Therefore, by orthonormality

$$\langle \mathbf{w}, \mathbf{v}_i \rangle = c_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2\langle \mathbf{v}_2, \mathbf{v}_i \rangle + \cdots + c_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle = c_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle = c_i$$

Recall: When we want to find coord., we have to set up matrix and solve the system of equations. For orthonormal basis, the process is much simpler - just take the inner product. With this advantages, we will look at a procedure for finding such a basis. This procedure is called Gram-Schmidt orthonormalization process

(4) **Gram-Schmidt orthonormalization process**: Let $B = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis for an inner product space of V.

$$\mathbf{w}_1 = \mathbf{v}_1$$
$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1$$
$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2$$
$$\vdots$$
$$\mathbf{w}_n = \mathbf{v}_n - \frac{\langle \mathbf{v}_v, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \cdots - \frac{\langle \mathbf{v}_n, \mathbf{w}_{n-1} \rangle}{\langle \mathbf{w}_{n-1}, \mathbf{w}_{n-1} \rangle}\mathbf{w}_{n-1}$$

Note that the set $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ is an orthogonal set. Now, we need to normalize it.
Let $\mathbf{u}_i = \dfrac{\mathbf{w}_i}{\|\mathbf{w}_i\|}$, then $B' = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is an orthonormal basis for V.

(5) Ex:

$$B = \{\underbrace{(1,1,0)}_{\mathbf{v}_1}, \underbrace{(1,2,0)}_{\mathbf{v}_2}, \underbrace{(0,1,2)}_{\mathbf{v}_3}\} \text{ is a basis for } \mathbb{R}^3$$

Applying Gram-Schmidt to B relative to Euclidean inner product.

- $\mathbf{w}_1 = \mathbf{v}_1 = (1,1,0)$
- $\mathbf{w}_2 = \mathbf{v}_2 - \dfrac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 = (1,2,0) - \dfrac{3}{2}(1,1,0) = \big(-\dfrac{1}{2}, \dfrac{1}{2}, 0\big)$
- $\mathbf{w}_3 = \mathbf{v}_3 - \dfrac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \dfrac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 = (0,1,2) - \dfrac{1}{2}(1,1,0) - \dfrac{1/2}{1/2}\big(-\dfrac{1}{2}, \dfrac{1}{2}, 0\big) = (0,0,2).$
- Normalize :

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{(1,1,0)}{\sqrt{2}} = \Big(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\Big)$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{(-\frac{1}{2}, \frac{1}{2}, 0)}{\sqrt{\frac{1}{4} + \frac{1}{4}}} = \Big(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\Big)$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{(0,0,2)}{2} = (0,0,1)$$

$U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.
To find the coordinates (or the coefficients) of $\mathbf{v} = (1,2,3)$ with respect to this basis, we only need to compute:

$$c_1 = \langle \mathbf{v}, \mathbf{u}_1 \rangle = \frac{3\sqrt{2}}{2}, \qquad c_2 = \langle \mathbf{v}, \mathbf{u}_2 \rangle = \frac{\sqrt{2}}{2}, \qquad c_3 = \langle \mathbf{v}, \mathbf{u}_3 \rangle = 3$$

This means, $\mathbf{v} = \dfrac{3\sqrt{2}}{2}\mathbf{u}_1 + \dfrac{\sqrt{2}}{2}\mathbf{u}_2 + 3\mathbf{u}_3$.