Orthogonal matrix: is a square matrix A with property $A^{-1} = A^T$.

Therefore, we can say A is orthogonal iff $A^T A = I$ or $AA^T = I$.

EX.: $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$ is an orthogonal matrix (*verify!*). What is A^{-1}?

Theorem: The following are equivalent for a $n \times n$ matrix A

- A is orthogonal
- Row vectors of A form an orthonormal set in $\mathbb{R}^n$ with the Euclidean inner product.
- Column vectors of A form an orthonormal set in $\mathbb{R}^n$ with the Euclidean inner product.

Theorem:

(1) The inverse of an orthogonal matrix is orthogonal.
(2) A product of orthogonal matrices is orthogonal.
(3) If A is orthogonal, then $|A| = \pm 1$.

Review: Coordinate relative to a basis.

Let $B = \{v_1, \ldots, v_n\}$ be a basis for a vector space V and $\mathbf{x}$ is a vector in V. Then $\mathbf{x}$ can be expressed uniquely as

$$\mathbf{x} = c_1 v_1 + \cdots + c_n v_n$$

The scalars $c_1, \ldots, c_n$ are called coordinate of $\mathbf{x}$ relative to basis B. We denote

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \qquad \text{as coordinate vector of } \mathbf{x} \text{ relative to } B.$$

<u>Ex:</u> Let B be the standard basis and $B' = \{(1,0,1), (0,-1,2), (2,3,-5)\}$ be a basis in $\mathbb{R}^3$. Given that $\mathbf{x} = (1,2,-1)$, find $[\mathbf{x}]_B$ and $[\mathbf{x}]_{B'}$.

$$(1,2,-1) = 1(1,0,0) + 2(0,1,0) + (-1)(0,0,1) \implies [\mathbf{x}]_B = (1,2,-1)$$
$$(1,2,-1) = c_1(1,0,1) + c_2(0,-1,2) + c_3(2,3,-5)$$
$$\implies \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \ (\star) \implies \begin{matrix} c_1 = 5 \\ c_2 = -8 \\ c_3 = -2 \end{matrix} \implies [\mathbf{x}]_{B'} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

The procedure demonstrated above is called <u>change of basis</u>. That is, given the coordinate of a vector relative to one basis B, find the coordinate relative to other basis B'.

Note that $(\star)$ can be written as:

$$P[x]_{B'} = [x]_B.$$

P is called *transition matrix from B' to B,* denoted as $P_{B \leftarrow B'}$.

Theorem:

$$[\mathbf{x}]_{B'} = P_{B' \leftarrow B}[\mathbf{x}]_B, \quad \text{and} \quad P_{B' \leftarrow B} = P^{-1}_{B \leftarrow B'}$$

Finding $P_{B \leftarrow B'}$: Use Gauss-Jordan elimination on

$$\begin{bmatrix} B & | & B' \end{bmatrix} \implies \begin{bmatrix} I_n & | & P \end{bmatrix}$$

Ex1: Find Transition matrix from B' to B for $\mathbb{R}^2$ where $B = \{(-3,2),(4,-2)\}$ and $B' = \{(-1,2),(-2,2)\}$.

$$[B \mid B'] = \begin{bmatrix} -3 & 4 & \mid & -1 & -2 \\ 2 & -2 & \mid & 2 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & \mid & 3 & 2 \\ 0 & 1 & \mid & 2 & 1 \end{bmatrix}$$

Thus, $P_{B \leftarrow B'} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ and $P_{B' \leftarrow B} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$.

Ex2: Let $V = P_2$ and consider bases $B = \{x^2, x, 1\}$, $S = \{1, 2x, x^2+4\}$, and $T = \{x+2, x^2-1, 2x\}$ for V.

$$[S \mid B] = \begin{bmatrix} 0 & 0 & 1 & \mid & 1 & 0 & 0 \\ 0 & 2 & 0 & \mid & 0 & 1 & 0 \\ 1 & 0 & 4 & \mid & 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} & \mid & -4 & 0 & 1 \\ I & \mid & 0 & \frac{1}{2} & 0 \\ & \mid & 1 & 0 & 0 \end{bmatrix} = [I_3 \mid P_{S \leftarrow B}]$$

$$[T \mid B] = \begin{bmatrix} 0 & 1 & 0 & \mid & 1 & 0 & 0 \\ 1 & 0 & 2 & \mid & 0 & 1 & 0 \\ 2 & -1 & 0 & \mid & 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} & \mid & 1/2 & 1 & -1/2 \\ I & \mid & 1 & 0 & 0 \\ & \mid & -1/4 & 1/2 & -1/4 \end{bmatrix} = [I_3 \mid P_{T \leftarrow B}]$$

$$[T \mid S] = \begin{bmatrix} 0 & 1 & 0 & \mid & 0 & 0 & 1 \\ 1 & 0 & 2 & \mid & 0 & 2 & 0 \\ 2 & -1 & 0 & \mid & 1 & 0 & 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} & \mid & 1/2 & 0 & 5/2 \\ I & \mid & 0 & 0 & 1 \\ & \mid & -1/4 & 1 & -5/4 \end{bmatrix} = [I_3 \mid P_{T \leftarrow S}]$$

$$[S \mid T] = \begin{bmatrix} & \mid & 2 & -5 & 0 \\ I & \mid & 1/2 & 0 & 1 \\ & \mid & 0 & 1 & 0 \end{bmatrix} = [I_3 \mid P_{S \leftarrow T}]$$

Find coordinate of $f(x) = -3x^2 + x - 2$ relative to $S, B,\ B'$

$$[f]_B = [-3,\ 1,\ -2]^T.$$

$$[f]_S = P_{S \leftarrow B}[f]_B$$

$$[f]_T = P_{T \leftarrow B}[f]_B$$

$$[f]_T = P_{T \leftarrow S}[f]_S$$