7.1: Eigenvalues-Eigenvectors:

(1) Def. Let A be $n \times n$ matrix, if there exists a non-zero vector $\mathbf{x}$ in $\mathbb{R}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for any scalar λ, then λ is called an **eigenvalue** of A, and $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ.

(2) **How to find eigenvalues-eigenvectors**
   (a) Set up the characteristic equation $det(\lambda I - A) = 0$.
   (b) Solve for λ.
   (c) Plug λ into the system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ and solve for $\mathbf{x}$. The solution space of $(\lambda I - A)\mathbf{x} = \mathbf{0}$ is called eigenspace of A corresponding to λ.

(3) **Theorems/Properties:**
   (a) Eigenspace; i.e., the solution or null space of $(\lambda I - A)\mathbf{x} = 0$, is a subspace of $\mathbb{R}^n$.
   (b) If A is $n \times n$ triangular matrix, then eigenvalues of A are the entries on the main diagonal of A.
   (c) A and A^T has the same eigenvalues. However, they do not necessarily have the same corresponding eigenvectors.
   (d) Determinant of A is the product of all eigenvalues of A and trace of A is the sum of the eigenvalues.
   (e) If k is a positive integer, λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.
   (f) A square matrix A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

7.2: Diagonalization:

(1) **Similar matrices**: A matrix B is similar to matrix A if there is a nonsingular matrix P such that $B = P^{-1}AP$.

(2) **diagonalized**: A square matrix A diagonalizable if it is similar to a diagonal matrix.

(3) **Theorem**:
   (a) Similar matrices have the same eigenvalues.
   (b) A is diagonalizable iff A has n linearly independent eigenvectors. In this case, we have $D = P^{-1}AP$ where D is a diagonal matrix whose diagonal entries are eigenvalues of A and P is a matrix whose columns are corresponding eigenvectors of A.
   (c) If A has n distinct eigenvalues, then A is diagonalizable.
   Remark: In case eigenvalues of A are not distinct: if each λ_i with multiplicity k has k linearly independent vector(s), then A is diagonalizable.

(4) **How to diagonalize a matrix**:
   (a) *Step 1*: Find n linearly independent eigenvectors of A, say, $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$.
   (b) *Step 2*: Form the matrix P having $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ as its column vectors.
   (c) *Step 3*: The matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries, where λ_i is the eigenvalue corresponding to $\mathbf{p}_i$, for $i = 1, 2, \ldots, n$.

7.3: Orthogonal Diagonalization:

(1) **orthogonal matrix**: A nonsingular matrix A is called orthogonal matrix if $A^{-1} = A^T$; that is, $AA^T = I_n$.

(2) **Theorem:** Let A be an $n \times n$ symmetric matrix.
- All eigenvalues of A are real.
- A is diagonalizable.
- Eigenvectors that are associated with distinct eigenvalues of A are orthogonal.

(3) **Theorem:** Let A be an $n \times n$ matrix. The following are equivalent:
- A is orthogonally diagonalizable.
- A has an orthonormal set of n eigenvectors.
- A is symmetric.