§8.1: General linear transformation

(1) Def.: If $T : V \to W$ is a function from a vector space V into a vector space W, then T is called a **linear transformation** from V to W if for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V and all scalars c

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$
$$T(c\mathbf{u}) = cT(\mathbf{u}).$$

In the special case where $V = W$, the linear transformation $T : V \to V$ is called a **linear operator** on V.
Ex.: $T : V \to V$ defined by $T(\mathbf{v}) = k\mathbf{v},\ k > 0$.
Ex.: $T : P_3 \to P_4$ defined by $T(p(x)) = xp(x)$.
Ex.: $T : R^3 \to R^2$ defined by $T(x, y, z,) = (x, y)$ (projection)
Ex.: $T : R^3 \to R^3$ defined by $T(\mathbf{u}) = r\mathbf{u}$ (dilation/contraction)
Ex.: $T : R^2 \to R^2$ defined by $T(x, y) = (x, -y)$ (reflection)
Ex.: $T : R^2 \to R^2$ defined by $T(\mathbf{u}) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \mathbf{u}$ (rotation)

Note: composition of linear transformation is also a linear transformation with $(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u}))$ where $\mathbf{u}$ is a vector in V.

(2) Important theorems Let $T : V \to W$ be a linear transformation, $\mathbf{v}_1, \cdots, \mathbf{v}_k$ be vectors in V, and $c_1, \cdots, c_k$ be scalars.
- $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_kT(\mathbf{v}_k)$.
- $T(\mathbf{0_V}) = \mathbf{0_W}$.
  Corollary: If $T(\mathbf{0_V}) \neq \mathbf{0_W}$, then T is a nonlinear transformation.
- $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$
- There exists a unique $m \times n$ matrix A such that $T(x) = Ax$.
- If $B = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is a basis for V, then for any vector $\mathbf{u}$ in V, $T(\mathbf{u})$ is completely determined by $\{T(\mathbf{v}_1), \cdots, T(\mathbf{v}_k)\}$.

§8.2: Kernel and Range

(1) **Definition**:
If $T : V \to W$ is a linear transformation, then the set of vectors in V that T maps into $\mathbf{0}$ is called the **kernel** of T; it is denoted by $ker(T)$. The set of all vectors in W that are images under L of at least one vector in V is called the **range** of T; it is denoted by $Range(T)$.

If A is standard matrix of T, then $ker(T) = nullspace(A)$ and $range(T) = col(A)$.

Ex.: $T : V \to V$ defined by $T(\mathbf{v}) = \mathbf{0}$.

$$ker(T) = V,\ range(T) = 0$$

Ex.: $T : R^2 \to R^2$ defined by $T(x, y) = (2x - y, -8x + 4y)$.

$$ker(T) = \{(-1/2, 1)\},\ range(T) = \{(2, -8)\}$$

(2) **Theorem**:
If $T : V \to W$ is a linear transformation, then:
(a) The kernel of T is a subspace of V.
(b) The range of T is a subspace of W.

(3) **Definition**:
If $T : V \to W$ is a linear transformation, then the dimension of the range of T is called the **rank of** T and is denoted by rank(T); the dimension of the kernel is called the **nullity of** T

and is denoted by nullity(T).

(4) **Theorem**:
If A is an $m \times n$ matrix and $T_A : \mathbb{R}^n \to R^m$ is multiplication by A, then:
  (a) nullity(L_A)=nullity(A)
  (b) rank(L_A)=rank(A)

(5) **Dimension Theorem for Linear Transformations:**
If $T : V \to W$ is a linear transformation from an $n-$dimensional vector space V to a vector space W, then
$rank(T)+nullity(T)=n$

§8.3: Inverse Linear Transformation

(1) **Definition**:
A linear transformation $T : V \to W$ is said to be **one-to-one** if $T(\mathbf{u}) = T(\mathbf{v}) \implies \mathbf{u} = \mathbf{v}$.
A linear transformation $T : V \to W$ is said to be **onto** if $range(T) = W$.

(2) **Theorem**:
If $T : V \to W$ is a linear transformation, then the following are equivalent.
  (a) T is one-to-one.
  (b) The kernel of T contains only the zero vector; that is, $ker(T) = \{\mathbf{0}\}$.
  (c) $nullity(T)$=0

(3) **Theorem**:
If $T : V \to W$ is a linear transformation and $dim(V) = dim(W)$, then T is one-to-one iff T is onto.

(4) **Inverse Linear Transformations**:
If T is one-to-one, then each vector $\mathbf{v}$ in V has a unique images $\mathbf{v} = T(\mathbf{v})$ in $R(T)$

$$T^{-1}(T(\mathbf{v})) = T^{-1}(\mathbf{w}) = \mathbf{v}$$
$$T(T^{-1}(\mathbf{w})) = T(T\mathbf{v}) = \mathbf{w}$$