Math 521A

6.2 – Quotient Rings and Homomorphisms

Theorem (Theorem 6.8)

Let I be an ideal in a ring R. If

$$a + I = b + I \quad \text{and} \quad c + I = d + I \quad \text{in } R/I$$

then

$$(a + c) + I = (b + d) + I \quad \text{and} \quad ac + I = bd + I.$$

Important consequence of Theorem 6.8: The result of "adding" or "multiplying" two cosets is independent of the choice of the representatives of the cosets, which motivates the following:

Definition

Let I be an ideal in a ring R. Addition and multiplication in R/I are defined by:

$$(a + I) + (c + I) = (a + c) + I \quad (1)$$

$$(a + I)(c + I) = ac + I. \quad (2)$$

Example

Let $R = \mathbb{Z}_{10},\ I = \{0, 5\}$. Then

$$R/I = \{I, 1+I, 2+I, 3+I, 4+I\}.$$

$$(2+I)+(3+I) = I,\quad (2+I)(3+I) = 1+I.$$

Example

$R = \mathbb{Z}[x],\ I = \{\text{all polynomials in } R \text{ with even constant terms}\}$. Then $I = (2, x)$ and $R/I = \{I, 1+I\}$. The addition and multiplication tables for R/I are shown on the board.

Theorem (Theorem 6.9)

Let I be an ideal in a ring R. Then:

(1) *R/I is a ring with addition and multiplication of cosets as defined in (1) and (2), respectively, on slide #2.*

(2) *If R is commutative then R/I is a commutative ring.*

(3) *If R has an identity then so does the ring R/I.*

Definition

The ring R/I is called the quotient ring or factor ring of R by I.

Example

If $R = \mathbb{Z}$ and $I = (2)$ then $R/I = \{I,\ 1+I\}$ is a ring isomorphic to $\mathbb{Z}_2$.

Homomorphisms:

Theorem (Theorem 6.10)

Let $f : R \to S$ be a ring homomorphism and

$$K = \{r \in R \mid f(r) = 0_S\}.$$

Then K is an ideal in R, called the the kernel of f.

The proof is presented on the board.

Example

Let R be a ring and consider the map $E_0 : R[x] \to R$ given by $E_0(p(x)) = p(0)$. Then E_0 is a ring homomorphism with kernel

$$K = \{p(x) \in R[x] \mid p(0) = 0\}.$$

K is an ideal in R.

Theorem (Theorem 6.11)

Let $f : R \to S$ be a ring homomorphism with kernel K. Then $K = \{0_R\}$ if and only if f is injective (i.e., one-to-one).

For the proof of $(\Leftarrow)$, recall that $f(0_R) = 0_S$ (Theorem 3.10, Part (1)). Hence, if f is injective, its kernel consists of 0_R only. The other direction $(\Rightarrow)$ is proved on the board.

Theorem (Theorem 6.12)

Let I be an ideal in a ring R. Then the map

$$\begin{array}{rccc} \pi : & R & \longrightarrow & R/I \\ & r & \longmapsto & r + I \end{array}$$

is a surjective homomorphism with kernel I.

The map π is called the natural homomorphism from R to R/I. The proof of the theorem is shown on the board.

Theorem (Theorem 6.13 – The First Isomorphism Theorem)

Let $f : R \to S$ be a surjective homomorphism of rings with kernel K. Then the quotient ring R/I is isomorphic to S.

Proof.

Define $\begin{array}{rccc} \varphi: & R/K & \longrightarrow & S \\ & r+K & \longmapsto & f(r) \end{array}$ and show that:

1. φ is well defined.
2. $\varphi((r+K)+(s+K)) = \varphi(r+K)+\varphi(s+K)\ \forall r,s \in R$.
3. $\varphi((r+K)(s+K)) = \varphi(r+K)\varphi(s+K)\ \forall r,s \in R$.
4. Kernel of $\varphi = \{K\} = \{0_{R/K}\}$ (this proves that φ is injective).
5. φ is surjective.

□

Example

$\begin{array}{rccc} E_0: & \mathbb{R}[x] & \longrightarrow & \mathbb{R} \\ & p(x) & \longmapsto & p(0) \end{array}$ is a surjective homomorphism with kernel $K = \{p(x) \in \mathbb{R}[x] \,|\, p(0) = 0\}$. Hence, $\mathbb{R}[x]/K \cong \mathbb{R}$.

Example

$\begin{array}{rccc} f: & \mathbb{Z}[x] & \longrightarrow & \mathbb{Z}_2 \\ & p(x) & \longmapsto & [p(0)]_2 \end{array}$ is a surjective homomorphism with kernel $K = \{p(x) \in \mathbb{R}[x] \,|\, p(0) = \text{ even}\}$. Hence, $\mathbb{Z}[x]/K \cong \mathbb{Z}_2$.